International Journal of Theoretical Physics, Vol. 28, No. 1, 1989

Quantum Causal Structure and the Einstein–Podolsky–Rosen Experiment

László E. Szabó[1]

Received February 3, 1988

In earlier papers—independently of the EPR problem—the author suggested a causal structure, which was intrinsically based on quantum theory. In this paper the causal relations of the crucial EPR events are analyzed in the light of the new conception of causality.

1. INTRODUCTION

Quantum theory is a statistical theory which fundamentally differs from classical statistical physics. In quantum theory the physical events form a non-Boolean lattice isomorphic with the subspace lattice of a Hilbert space. Therefore, the quantum mechnical probability is a "probability measure" defined on a non-Boolean lattice of events. Quantum logic is just that very discipline which investigates this deep-rooted structure of physical events (propositions). This structure is *independent* of the probability distributions, i.e., it does not depend on the potential states of the system.

Similarly, the question of whether two events of space-time are causal or not can be answered on the strength of immanent structure of space-time *independently* of whether a real physical action propagation occurs in fact or not. On the other hand, it would be paradoxical if any physical action propagated between spatially separated regions.

In earlier papers (Szabó, 1986, 1987*a*) axioms were suggested for the causal structure of events, in which the subset lattice of space-time was replaced by the dual of the quantum lattice. It was called a quantum causal structure. I recall these axioms in Section 2. It turns out that the causal relations of physical events of the new causal structure can be quite different from that of the conventional one. Therefore, events which *can be* causally

[1]Institute for Theoretical Physics, Eötvös University, Budapest, Hungary.

separated in the classical theory *cannot be* causally separated in the quantum causal structures.

As is well known, the predictions of quantum theory in certain spin-correlation experiments contradict the conventional causal structure. In Section 3 I summarize this problem. In Section 4 the causal relations between the crucial EPR events are analyzed within the framework of a quantum causal structure.

It turns out that the nonlocal character of quantum mechanics is compatible with the quantum causal structures.

2. QUANTUM CAUSAL STRUCTURE

The primary object of the current formulation of general relativity is an "underlying set" on which the (causal, topological, geometrical) structure of space-time is defined. The elements of this set are usually called "events." However, on closer analysis the meaning of an event is rather vague, especially if we want to abstain from tautology: defining an event with reference to the space-time structure itself.

In quantum theory the accepted definition of a physical event is the following: A physical event means a possible result of a possible measurement or observation performed on a physical object. At first sight this definition is quite far from the notion of a space-time point as an event. However, each space-time point can be formulated in the language of physical observations. On the other hand, each physical event takes place somewhere in space-time.

In the present work the space-time structure is built up on the ground of the quantum lattice of events. As an initial step, one can generalize the axiomatic theory of a causal space by Kronheimer and Penrose (1967) to the case where the lattice of physical events is not Boolean but it is quantum lattice (Jauch, 1968). The starting point of the generalization is the reformulation of the axioms of Kronheimer and Penrose in terms of causal and chronological futures and pasts of the space-time subsets. Then we replace the subset lattice by the dual of a quantum lattice $(S, \wedge, \vee)$.

We define a quantum causal structure via two pairs of maps

$$J^{\pm}\colon \quad S \to S, \qquad I^{\pm}\colon \quad S \to S \tag{1}$$

with the following properties:

Q1. $A < J^{\pm}(A)$
Q2. $I^{\pm}(A) < J^{\pm}(A)$
Q3. $x < J^{\pm}(y)$ & $y < J^{\pm}(x) \Rightarrow x = y$
Q4. $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$
Q5. $J^{\pm}(A \vee B) = J^{\pm}(A) \vee J^{\pm}(B)$

Q6. $J^{\pm}(A \wedge B) < J^{\pm}(A) \wedge J^{\pm}(B)$
Q7. $I^{\pm}(A \vee B) = I^{\pm}(A) \vee I^{\pm}(B)$
Q8. $I^{\pm}(A \wedge B) < I^{\pm}(A) \wedge I^{\pm}(B)$
Q9. $J^{\pm}(I^{\pm}(A)) < I^{\pm}(A)$
Q10. $I^{\pm}(J^{\pm}(A)) < I^{\pm}(A)$
Q11. $x \not< I^{\pm}(x)$
Q12. $x < J^{+}(y) \Leftrightarrow y < J^{-}(x)$
Q13. $x < I^{+}(y) \Leftrightarrow y < I^{-}(x)$
Q14. $I^{\pm}(I) \neq \varnothing$

Here $A, B \in S$ and $x, y \in \mathscr{A}(S)$, and $\mathscr{A}(S)$, I, and $\varnothing$ denote the set of atoms, the maximal element, and the minimal element of S, respectively.

In a quantum causal structure we define the causality and chronology relations as follows:

$A <_c B$ if and only if $B < J^{+}(A)$ or $A < J^{-}(B)$
$A \ll B$ if and only if $B < I^{+}(A)$ or $A < I^{-}(B)$

Here $A, B \in S$. Two events A and B are said to be spatially separated if neither $A <_c B$ nor $B <_c A$ holds.

If S is a Boolean lattice, it can be represented by a suitable subset lattice and the quantum causality leads to the usual causality on an "underlying set" of Kronheimer and Penrose. It has to be remarked that we have good hopes of building up the whole space-time structure on this ground. For example, the equivalent of the Alexandrov topology is already known (Szabó, 1987*a*).

3. EPR EXPERIMENTS

Let us consider a spin-0 system which consists of two spin-1/2 particles. The spin part of the state vector is given by

$$\Psi = \frac{1}{\sqrt{2}}(u_{\mathbf{n}}^{+}(1) \otimes u_{\mathbf{n}}^{-}(2) - u_{\mathbf{n}}^{-}(1) \otimes u_{\mathbf{n}}^{+}(2)) \tag{2}$$

where $\sigma \cdot \mathbf{n} u_{\mathbf{n}}^{\pm}(1) = \pm u_{\mathbf{n}}^{\pm}(1)$, so that $u_{\mathbf{n}}^{\pm}(1)$ describes a state in which particle 1 has spin up or down, respectively, along the direction $\mathbf{n}$, and $u_{\mathbf{n}}^{\pm}(2)$ has a similar meaning concerning particle 2. We now assume that the two particles are isolated from each other, by separating them spatially. According to the classical space-time causality, any observation carried out on one of the particles cannot have any physical effect on the other particle.

After this separation the state vector of the whole system is still given by (2). Suppose that we measure the spin of particle 1 along the direction $\mathbf{a}$. *The outcome is not predetermined by the state vector* Ψ. But it determines entirely the outcome of measurement carried out on particle 2. If particle 1 is found to be in state $u_{\mathbf{a}}^{1}(1)$, then particle 2 will be found to be in state $u_{\mathbf{a}}^{-}(2)$ if the **a** component of its spin is measured. Likewise, the outcome of measurement performed on particle 2 determines the outcome of measurement performed on particle 1. This means that *we find a manifest correlation between the physical events corresponding to the subspaces generated by vectors* $u_{\mathbf{a}}^{+}(1)$ *and* $u_{\mathbf{a}}^{-}(2)$ *or* $u_{\mathbf{a}}^{-}(1)$ *and* $u_{\mathbf{a}}^{+}(2)$ *in spacelike-separated regions A and B.*

One possibility, it had been hoped, is to reinterpret quantum mechanics in terms of a statistical account of an underlying local hidden-variables theory. The value of these variables in space-time region A would locally *determine* the outcome of the measurement independently of anything performed in region B. However, Bell's theorem has shown that this cannot be done (Clauser and Shimony, 1978).

In the rest of this paper I am going to show that the direct causal effect between the EPR events is *not* excluded if we assume that the intrinsic causality of physical events is described by a quantum causal structure (over a non-Boolean lattice of events) rather than by the conventional relativity.

4. CAUSALITY OF THE EPR EVENTS IN THE QUANTUM CAUSAL STRUCTURES

Let us now consider the EPR experiment from the point of view of quantum logic. The spin part of the state space is $H^2 \otimes H^2$. Consider a finite sublattice (see Figure 1) of the subspace lattice generated by five atoms. These atoms are one-dimensional subspaces determined by the following vectors:

A $u_{\mathbf{a}}^{+}(1)\otimes u_{\mathbf{a}}^{-}(2)$
B $u_{\mathbf{a}}^{-}(1)\otimes u_{\mathbf{a}}^{+}(2)$
C $u_{\mathbf{a}}^{+}(1)\otimes u_{\mathbf{a}}^{+}(2)$
D $u_{\mathbf{a}}^{-}(1)\otimes u_{\mathbf{a}}^{-}(2)$
E $u_{\mathbf{a}}^{+}(1)\otimes u_{\mathbf{a}}^{-}(2)-u_{\mathbf{a}}^{-}(1)\otimes u_{\mathbf{a}}^{+}(2)$

The interesting EPR events correspond to the following elements of the lattice:

particle 1 has spin "up" along direction **a**	b
particle 1 has spin "down" along direction **a**	e
particle 2 has spin "up" along direction **a**	d
particle 2 has spin "down" along direction **a**	c

The following question arises: are the events b and c as well as e and d spatially separated or not? If the answer is no, *the direct causal effects are*

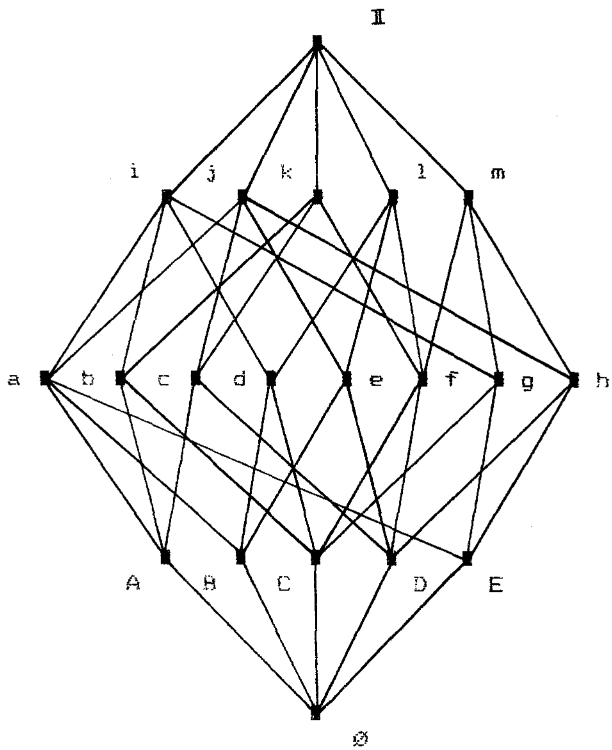

Fig. 1. The smallest non-Boolean model quantum lattice describing the EPR events.

not excluded, hence the quantum causal structure resolves the EPR paradox. If it is yes, the paradox remains unsolved.

The quantum causal structure is based on the dual of the quantum lattice of events. Denote by S the dual of the model quantum lattice in question (see Figure 2). The events b, c, e, d mentioned above correspond to the elements 12, 11, 9, and 10 of this lattice.

Of course, there can exist a number of quantum causal structures of S satisfying axioms Q1, . . . , Q14. Analogically to conventional relativity, where on a given manifold one can also introduce many different metrics satisfying the conditions required for a space-time, I have examined by computer (Szabó, 1987*b*) all possible quantum causal structures on lattice S, i.e., all quadruples $(J^{\pm}, I^{\pm})$ of maps from S to itself (from the 80^{20} ones) which satisfy axioms Q1, . . . , Q4. There exist 48 quantum causal structures on the lattice S (see Table I). Considering these causal structures, one finds the following surprising result. In each quantum causal structure the events 12 and 11 as well as 9 and 10 are *not* spatially separated. Note that, for

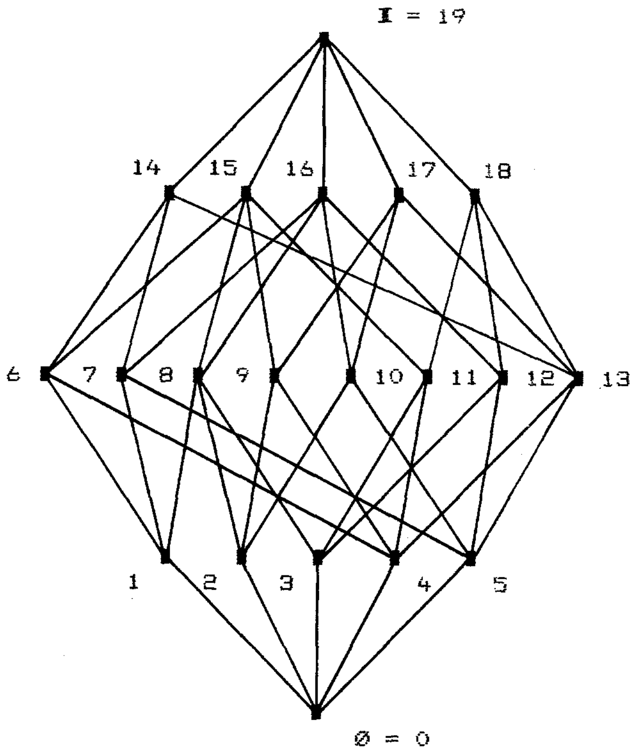

Fig. 2. Lattice S is the dual of the model quantum lattice.

instance, the relation $12 < J^+(11)$ is not inconsistent with $11 < J^+(12)$, since the events 12, 11, 9, and 10 are not atoms of the lattice S. The same situation can appear in the conventional theory of causality for subsets of space-time containing more than one point.

5. CONCLUSIONS

Having assumed that the intrinsic causality of physical events is described by a quantum causal structure, we have found that the EPR paradox is resolved within the framework of a finite model quantum lattice. According to the following theorem, this result holds for a large family of model quantum lattices.

Theorem. Denote by $\tilde{S}$ a finite sublattice of the dual subspace lattice of the Hilbert space $H^2 \otimes H^2$. Let S be a sublattice of $\tilde{S}$ such that $I = \tilde{I}$ and $O = \tilde{O}$. Suppose $\tilde{S}$ is equipped with a quantum causal structure $(\tilde{J}^{\pm}, \tilde{I}^{\pm})$.

Table I

No.	A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	J+	1	2	3	4	13	6	14	8	9	17	11	18	13	14	15	19	17	18	19
	I+	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
	J−	1	2	3	13	5	14	7	8	17	10	18	12	13	14	19	16	17	18	19
	I−	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
2	J+	1	2	3	4	16	6	16	8	9	16	11	16	19	19	15	16	19	19	19
	I+	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
	J−	7	10	12	4	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I−	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
3	J+	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I+	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
	J−	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I−	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
4	J+	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I+	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
	J−	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I−	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
5	J+	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I+	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
	J−	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I−	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
6	J+	1	2	3	13	5	14	7	8	17	10	18	12	13	14	19	16	17	18	19
	I+	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
	J−	1	2	3	4	13	6	14	8	9	17	11	18	13	14	15	19	17	18	19
	I−	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
7	J+	1	2	3	15	5	15	7	8	15	10	15	12	19	19	15	16	19	19	19
	I+	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
	J−	6	9	11	4	5	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I−	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
8	J+	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I+	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
	J−	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I−	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13
9	J+	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I+	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
	J−	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I−	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
10	J+	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I+	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
	J−	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I−	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
11	J+	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I+	0	0	0	8	15	8	15	0	8	15	8	15	15	15	8	15	15	15	15
	J−	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I−	13	13	13	5	0	13	13	13	13	13	13	13	5	13	13	13	13	13	13
12	J+	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I+	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
	J−	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I−	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13

Table I. Continued

No.	A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
13	J+	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I+	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
	J-	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
14	J+	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I+	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
	J-	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
15	J+	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I+	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
	J-	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I-	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
16	J+	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I+	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
	J-	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I-	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
17	J+	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I+	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
	J-	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I-	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
18	J+	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I+	0	0	0	16	8	16	8	0	16	8	16	8	16	16	16	8	16	16	16
	J-	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	13	13	13	0	4	13	13	13	13	13	13	13	4	13	13	13	13	13	13
19	J+	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I+	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
	J-	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
20	J+	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I+	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
	J-	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13
21	J+	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I+	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
	J-	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I-	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
22	J+	6	9	11	4	5	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I+	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
	J-	1	2	3	15	5	15	7	8	15	10	15	12	19	19	15	16	19	19	19
	I-	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
23	J+	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I+	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	J-	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I-	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
24	J+	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I+	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
	J-	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I-	0	0	0	8	0	8	0.	0	8	0	8	0	8	8	8	0	8	8	8

Table I. Continued.

No.	A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
25	J+	6	9	11	4	13	6	14	15	9	17	11	18	13	14	15	19	17	18	19
	I+	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
	J-	1	2	3	19	5	19	7	8	19	10	19	12	19	19	19	16	19	19	19
	I-	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
26	J+	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I+	4	4	4	0	15	4	15	4	4	15	4	15	15	15	4	15	15	15	15
	J-	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I-	5	5	5	16	0	16	5	5	16	5	16	5	16	16	16	5	16	16	16
27	J+	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I+	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	J-	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I-	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
28	J+	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I+	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
	J-	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I-	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
29	J+	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I+	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
	J-	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I-	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
30	J+	7	10	12	4	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I+	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
	J-	1	2	3	4	16	6	16	8	9	16	11	16	19	19	15	16	19	19	19
	I-	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
31	J+	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I+	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	J-	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I-	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
32	J+	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I+	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
	J-	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I-	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
33	J+	7	10	12	13	5	14	7	16	17	10	18	12	13	14	19	16	17	18	19
	I+	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
	J-	1	2	3	4	19	6	19	8	9	19	11	19	19	19	15	19	19	19	19
	I-	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
34	J+	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I+	5	5	5	16	0	16	5	5	16	5	16	5	16	16	16	5	16	16	16
	J-	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I-	4	4	4	0	15	4	15	4	4	15	4	15	15	15	4	15	15	15	15
35	J+	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I+	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	J-	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I-	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
36	J+	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I+	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
	J-	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I-	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4

Table I. Continued.

No.	A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
37	J+	7	10	12	19	5	19	7	16	19	10	19	12	19	19	19	16	19	19	19
	I+	0	0	0	5	0	5	0	0	5	0	5	0	5	5	5	0	5	5	5
	J-	6	9	11	4	19	6	19	15	9	19	11	19	19	19	15	19	19	19	19
	I-	0	0	0	0	4	0	4	0	0	4	0	4	4	4	0	4	4	4	4
38	J+	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13
	J-	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I-	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
39	J+	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
	J-	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I-	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8
40	J+	14	17	18	4	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
	J-	1	2	3	15	16	15	16	8	15	16	15	16	19	19	15	16	19	19	19
	I-	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
41	J+	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	13	13	13	0	4	13	13	13	13	13	13	13	4	13	13	13	13	13	13
	J-	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I-	0	0	0	16	8	16	8	0	16	8	16	8	16	16	16	8	16	16	16
42	J+	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13
	J-	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I-	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
43	J+	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	J-	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I-	0	0	0	16	0	16	0	0	16	0	16	0	16	16	16	0	16	16	16
44	J+	14	17	18	4	13	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	4	4	4	0	0	4	4	4	4	4	4	4	0	4	4	4	4	4	4
	J-	1	2	3	19	16	19	16	8	19	16	19	16	19	19	19	16	19	19	19
	I-	0	0	0	8	0	8	0	0	8	0	8	0	8	8	8	0	8	8	8
45	J+	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	13	13	13	5	0	13	13	13	13	13	13	13	5	13	13	13	13	13	13
	J-	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I-	0	0	0	8	15	8	15	0	8	15	8	15	15	15	8	15	15	15	15
46	J+	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	13	13	13	0	0	13	13	13	13	13	13	13	0	13	13	13	13	13	13
	J-	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I-	0	0	0	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	8
47	J+	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	J-	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I-	0	0	0	0	15	0	15	0	0	15	0	15	15	15	0	15	15	15	15
48	J+	14	17	18	13	5	14	14	19	17	17	18	18	13	14	19	19	17	18	19
	I+	5	5	5	0	0	5	5	5	5	5	5	5	0	5	5	5	5	5	5
	J-	1	2	3	15	19	15	19	8	15	19	15	19	19	19	15	19	19	19	19
	I-	0	0	0	0	8	0	8	0	0	8	0	8	8	8	0	8	8	8	8

There exists a quantum causal structure on lattice S such that for any $A, B \in S$ we have

$$A < J^{\pm}(B) \Rightarrow A < \tilde{J}^{\pm}(B) \tag{3}$$

and

$$A < I^{\pm}(B) \Rightarrow A < \tilde{I}^{\pm}(B) \tag{4}$$

(The proof is given in the Appendix.) This means that if one finds a quantum causal structure on an arbitrary larger (but finite) model lattice, the events 12 and 11 as well as 9 and 10 cannot be spatially separated, However, the existence of quantum causal structure on an arbitrary large event lattice has not been proved yet.

APPENDIX

First we shall prove the following.

Lemma. The quadruple of maps $(J^{\pm}, I^{\pm})$ defined by

$$\begin{aligned} J^{\pm}:\quad & A \in S \mapsto J^{\pm}(A) := \bigvee_{\substack{a \in \tilde{J}^{\pm}(A) \\ a \in S}} a \in S \\ I^{\pm}:\quad & A \in S \mapsto I^{\pm}(A) := \bigvee_{\substack{a \in \tilde{I}^{\pm}(A) \\ a \in S}} a \in S \end{aligned} \tag{5}$$

satisfies axioms Q1, ..., Q14 whenever $\tilde{S}$ contains only one more atom than S. The properties (3) and (4) hold trivially.

Proof of the Lemma. Denote by $\tilde{a}$ this new atom. It is obvious that $J^{\pm}(A)$ is the greatest of elements in S which are smaller than $\tilde{J}^{\pm}(A)$, and the same holds in respect to $I^{\pm}(A)$. Let us prove now that the maps $J^{\pm}$, $I^{\pm}$ satisfy the axioms Q1, ..., Q14.

Q1. $A < \tilde{J}^{\pm}(A)$. Since $J^{\pm}(A)$ is the greatest element in S which is smaller than $\tilde{J}^{\pm}(A)$, we have $A < J^{\pm}(A)$. ■

Q2. $I^{\pm}(A) < \tilde{I}^{\pm}(A) < \tilde{J}^{\pm}(A) \Rightarrow I^{\pm}(A)$. ■

Q3. $x < J^{\pm}(y)$ and $y < J^{\pm}(x) \Rightarrow x < \tilde{J}^{\pm}(y)$ and $y < \tilde{J}^{\pm}(x) \Rightarrow x = y$. ■

Q4. $J^{\pm}(J^{\pm}(A)) < \tilde{J}^{\pm}(J^{\pm}(A)) < \tilde{J}^{\pm}(\tilde{J}^{\pm}(A)) = \tilde{J}^{\pm}(A) \Rightarrow J^{\pm}(J^{\pm}(A)) < J^{\pm}(A)$. From Q1 it follows that $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$. ■

Q5.
$$\begin{aligned} & J^{\pm}(A) \vee J^{\pm}(B) < \tilde{J}^{\pm}(A) \vee \tilde{J}^{\pm}(B) = \tilde{J}^{\pm}(A \vee B) \\ & \Rightarrow J^{\pm}(A) \vee J^{\pm}(B) < J^{\pm}(A \vee B) \end{aligned} \tag{6}$$

At the same time

$$J^{\pm}(A \vee B) < \tilde{J}^{\pm}(A \vee B) = \tilde{J}^{\pm}(A) \vee \tilde{J}^{\pm}(B)$$

If $\tilde{J}^{\pm}(A)$ and $\tilde{J}^{\pm}(B)$ are contained in $S \subset \tilde{S}$, then $J^{\pm}(A) = \tilde{J}^{\pm}(A)$ and $J^{\pm}(B) = \tilde{J}^{\pm}(B)$; therefore,

$$J^{\pm}(A \vee B) < J^{\pm}(A) \vee J^{\pm}(B) \tag{7}$$

If $\tilde{J}^{\pm}(A)$ and (or) $\tilde{J}^{\pm}(B)$ are not contained in S, then $\tilde{J}^{\pm}(A) = J^{\pm}(A) \vee \tilde{a}$ and (or) $\tilde{J}^{\pm}(B) = J^{\pm}(B) \vee \tilde{a}$. Thus,

$$\tilde{J}^{\pm}(A) \vee \tilde{J}^{\pm}(B) = [J^{\pm}(A) \vee J^{\pm}(B)] \vee \tilde{a} \tag{8}$$

On the other hand,

$$[J^{\pm}(A) \vee J^{\pm}(B)] \wedge \tilde{a} = \varnothing \tag{9}$$

Taking into account that $\tilde{S}$ is absolute atomic (Fáy and Tőrös, 1978), i.e., for any element A and for any atoms p from $A \vee p = \varnothing$ it follows that A precedes element $A \vee p$, from (8) and (9) it follows that $J^{\pm}(A) \vee J^{\pm}(B)$ precedes $\tilde{J}^{\pm}(A) \vee \tilde{J}^{\pm}(B) = \tilde{J}^{\pm}(A \vee B)$; therefore,

$$J^{\pm}(A \vee B) < J^{\pm}(A) \vee J^{\pm}(B) \tag{10}$$

From (6), (7), and (10) we have

$$J^{\pm}(A \vee B) = J^{\pm}(A) \vee J^{\pm}(B) \quad \blacksquare$$

Q6. $J^{\pm}(A \wedge B) < \tilde{J}^{\pm}(A \wedge B) < \tilde{J}^{\pm}(A) \vee \tilde{J}^{\pm}(B) \Rightarrow J^{\pm}(A \wedge B) < J^{\pm}(A) \wedge J^{\pm}(B)$. $\blacksquare$

Q7. Similar to Q5.

Q8. Similar to Q6.

Q9. $J^{\pm}(I^{\pm}(A)) < \tilde{J}^{\pm}(I^{\pm}(A)) < \tilde{J}^{\pm}(\tilde{I}^{\pm}(A)) < \tilde{I}^{\pm}(A)) \Rightarrow J^{\pm}(I^{\pm}(A)) < I^{\pm}(A)$. $\blacksquare$

Q10. Similar to Q9.

Q11. $x < I^{\pm}(y) \Rightarrow x < \tilde{I}^{\pm}(x)$, which is not true.

Q12. $x < J^{+}(y) < \tilde{J}^{+}(y) \Rightarrow y < \tilde{J}^{-}(x) \Rightarrow y < J^{-}(x)$. $\blacksquare$

Q13. Similar to Q12.

Q14. Indirectly, suppose $\tilde{I}^{\pm}(\mathbb{I}) \neq \varnothing$ but $I^{\pm}(\mathbb{I}) = \varnothing$. Accordingly, we have

$$\text{not } \tilde{a} < \tilde{I}^{\pm}(\tilde{a}) \Rightarrow \tilde{I}^{\pm}(\tilde{a}) \in S \Rightarrow \tilde{I}^{\pm}(\mathbb{I}) \neq \tilde{a} \Rightarrow$$

$$I^{\pm}(\mathbb{I}) \neq \varnothing \Rightarrow \tilde{I}(\tilde{a}) = \varnothing$$

and for each $a \in \mathbb{A}(S)$

$$\tilde{I}^{+}(a) = \begin{cases} \tilde{a} \\ \varnothing \end{cases}$$

If $\tilde{I}^{+}(a)=\tilde{a}$, then $a<\tilde{I}^{-}(\tilde{a})$; therefore, $\tilde{I}^{\pm}(a)=\varnothing$ for each $a\in\mathbb{A}(S)$. In this case

$$\tilde{I}^{-}(a)=\varnothing \qquad \text{for any} \quad a\in\mathbb{A}(\tilde{S})$$

Consequently, $\tilde{I}^{-}(\mathbb{I})=\varnothing$. ■

Proof of the Theorem. Let $\tilde{S}=S_1\supset S_2\supset S_3\supset\cdots\supset S_n=S$ be a sequence of sublattices such that S_{i+1} contains one atom less than S_i. From the lemma it follows that on each S_i there exists a quantum causal structure $(J^{\pm}, I^{\pm})$ such that

$$A<J^{\pm}(B)\Rightarrow A<J^{\pm}_{n-1}(B)\Rightarrow\cdots\Rightarrow A<J^{\pm}_{2}(B)\Rightarrow A<\tilde{J}^{\pm}(B)$$

and

$$A<I^{\pm}(B)\Rightarrow A<I^{\pm}_{n-1}(B)\Rightarrow\cdots\Rightarrow A<I^{\pm}_{2}(B)\Rightarrow A<\tilde{I}^{\pm}(B) \quad ■$$

ACKNOWLEDGMENTS

I am indebted to P. Hraskó, Z. Perjés, and L. Szabados for fruitful discussions.

REFERENCES

Clauser, J., and Shimony, A. (1978). *Reports on Progress in Physics*, **41**, 1881.

Fáy, Gy., and Tőrös, R. (1978). *Kvantumlogika*, Gondolat, Budapest.

Jauch, J. M. (1968). *Foundation of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kronheimer, E. H., and Penrose, R. (1967). *Proceedings of the Cambridge Philosophical Society*, **63**, 418.

Szabó, L. (1986). *Journal of Mathematical Physics*, **27**, 2709.

Szabó, L. (1987*a*). *International Journal of Theoretical Physics*, **26**, 833.

Szabó, L. (1987*b*). Exploration of quantum causal structures on a finite lattice, ITP Budapest, Report 451.